Lecture 10: Some Important Discrete Probability Distributions (Part 1)

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Statistics (MAT1003)

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Outline

1. **Binomial, Multinomial Distributions**
   - Preliminaries
   - Binomial Distribution
   - Multinomial Distribution
2. **Hypergeometric Distribution**
3. **Geometric Distribution**

book: Sections 5.3, 5.4

And now . . .

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- Preliminaries
- Binomial Distribution
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2 Hypergeometric Distribution

3 Geometric Distribution

Bernoulli process

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We say: X has a binomial distribution with parameters $n = 10$ (the number of trials) and $p = 0.25$ (success probability)

Binomial distribution

If we do n independent trials, where each time the success probability is $p \in [0,1]$, and we define X to be the number of successes, then $X \sim B(n,p)$, and hence

$$P(X=k) = \underbrace{\binom{n}{k}}_{\text{pick } k \text{ out of } n} \cdot \underbrace{p^k}_{k \text{ successes, each with probability } p} \cdot \underbrace{(1-p)^{n-k}}_{n-k \text{ failures}}, \quad k = 0,1,2,\ldots,n$$

Moreover, $X \sim B(n,p) \Rightarrow E(X) = np, V(X) = np(1-p)$

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Appendix 1 (pp. 726-731): Table of $P(X \leq k)$
Apply: $P(X=k) = P(X \leq k) - P(X \leq k-1)$

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Apply: $P(X = k) = P(X \leq k) - P(X \leq k - 1)$

In Example 1 (a)

$P(1 \leq X \leq 3) = P(X \leq 3) - P(X \leq 0) = 0.7759 - 0.0563 = 0.7196$
If (n, p) is not in the table, then make the calculation yourself!

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- $X \sim B(n, p)$ with $n = 7$, $p = \frac{3}{10} = 0.3$
- $P(X = 4) = \binom{7}{4} \cdot (0.3)^4 \cdot (0.7)^3 \approx 0.0972$

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- Define $X := \sharp$ blue marbles, $Y := \sharp$ red marbles
- $X \sim B(7, 0.3)$, $Y \sim B(7, 0.2)$ $\Rightarrow X + Y \sim B(n, p)$ with $n = 7$, $p = \frac{5}{10} = 0.5$

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- $P(X = 2, Y = 1) = P(X = 2 | Y = 1) \cdot P(Y = 1)$
  $= P(X = 2 | Y = 1) \cdot \begin{pmatrix} 7 \\ 1 \end{pmatrix} \left(\frac{2}{10}\right)^1 \left(\frac{8}{10}\right)^6 = \ldots$

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- For $Y = 1$: $\left\{ \begin{array}{l} \text{1. 6 trials left, still independent} \\ \text{2. } P(\{\text{blue}\}) = \frac{3}{8} \text{ all the time} \end{array} \right\}$ *Bernoulli process* with $n = 6, p = \frac{3}{8}$

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$$\dots = \begin{pmatrix} 6 \\ 2 \end{pmatrix} \cdot \left(\frac{3}{8}\right)^2 \cdot \left(\frac{5}{8}\right)^4 \cdot \begin{pmatrix} 7 \\ 1 \end{pmatrix} \left(\frac{2}{10}\right)^1 \left(\frac{8}{10}\right)^6 = \frac{189}{1600} \approx 0.118$$

Observation

In the previous example ...

$$P(\overbrace{X=2}^{B},\overbrace{Y=1}^{R}) = \binom{6}{2}\cdot\left(\frac{3}{8}\right)^2\cdot\left(\frac{5}{8}\right)^4\cdot\binom{7}{1}\left(\frac{2}{10}\right)^1\left(\frac{8}{10}\right)^6$$

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$$= \binom{7}{1}\cdot\binom{6}{2}\cdot\frac{3^2\cdot 5^4}{8^6}\cdot\frac{2^1\cdot 8^6}{10^7}$$

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$$\underset{=}{\text{book}} \underbrace{\binom{7}{1,2,4}}_{\substack{\sharp\,\text{divisions}\\ \text{of group of size 7}\\ \text{into groups of}\\ \text{size } 1,2, \text{ and } 4}} \underbrace{\left(\frac{3}{10}\right)^2}_{\substack{P(\{blue\})\\ 2\times}} \cdot \underbrace{\left(\frac{5}{10}\right)^4}_{\substack{P(\{black\})\\ 4\times}} \cdot \underbrace{\left(\frac{2}{10}\right)^1}_{\substack{P(\{red\})\\ 1\times}}$$

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$\Rightarrow$ "Trinomial" distribution: $\left\{\begin{array}{l} \text{1. n independent trials} \\ \text{2. 3 possible outcomes per trial with constant probabilities} \end{array}\right.$

$\Rightarrow$ corresponding process very similar to Bernoulli process

Multinomial distribution - formulation

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$X_1 : \sharp$ red, $X_2 : \sharp$ blues, $X_3 : \sharp$ blacks, $p_1 = \frac{2}{10}$, $p_2 = \frac{3}{10}$, $p_3 = \frac{5}{10}$

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$$P(x_1 = k_1, x_2 = k_2, \dots, x_m = k_m) = \frac{n!}{k_1! \cdot k_2! \dots \cdot k_m!} \cdot p_1^{k_1} \cdot p_2^{k_2} \cdot \dots \cdot p_m^{k_m}$$

with $\sum_{i=1}^{m} p_i = 1$, $\sum_{j=1}^{m} k_j = n$

Example 2(c)

A bag contains 2 red, 3 blue, and 5 black marbles. We pick out 1 at random, write down the color, and then put it back. We repeat the procedure 7 times. What is the probability of picking 2 red, 2 blue, and 3 black marbles?
$X_1 : \sharp$ red, $X_2 : \sharp$ blues, $X_3 : \sharp$ blacks, $p_1 = \frac{2}{10}$, $p_2 = \frac{3}{10}$, $p_3 = \frac{5}{10}$
$P(X_1 = 2, X_2 = 2, X_3 = 3) =$

Multinomial distribution - formulation

If we have more than 2 possible outcomes, we call the distribution multinomial.
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And now . . .

1. Binomial, Multinomial Distributions
   - Preliminaries
   - Binomial Distribution
   - Multinomial Distribution
2. **Hypergeometric Distribution**
3. Geometric Distribution

Example 2(d)

A bag contains 2 red, 3 blue, and 5 black marbles. What if we take the marbles without replacement? Let's pick 4

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A bag contains 2 red, 3 blue, and 5 black marbles. What if we take the marbles without replacement? Let's pick 4

X : ♯ blues, Y : ♯ reds

$$P(X = 2) = \qquad , \; P(Y = 1) =$$

$$P(X + Y = 3) =$$

$$P(X = 2, Y = 1) =$$

Example 2(d)

A bag contains 2 red, 3 blue, and 5 black marbles. What if we take the marbles without replacement? Let's pick 4

X : ♯ blues, Y : ♯ reds

$$P(X=2)=\frac{\binom{3}{2}\binom{7}{2}}{\binom{10}{4}}= \quad , P(Y=1)=$$

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$$P(X+Y=3) = \frac{\binom{5}{3}\binom{5}{1}}{\binom{10}{4}} = 5/21$$

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Corresponding distribution: Hypergeometric (HW: derive general formula + try odd exercises from Sections 5.3 & 5.4)

And now . . .

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Example 1 (b)

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25 % of the families in a village have a subscription to a local newspaper

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25 % of the families in a village have a subscription to a local newspaper
Define Z: $\sharp$ families we have to ask such that the last family we ask is the first one that has a subscription

$$P(Z = 1) = 0.25$$

Example 1 (b)

25 % of the families in a village have a subscription to a local newspaper
Define Z: $\sharp$ families we have to ask such that the last family we ask is the first one that has a subscription

$$P(Z = 1) = 0.25$$
$$P(Z = 2) =$$

Example 1 (b)

25 % of the families in a village have a subscription to a local newspaper
Define Z: $\sharp$ families we have to ask such that the last family we ask is the first one that has a subscription

$$P(Z = 1) = 0.25$$
$$P(Z = 2) = 0.75 \cdot 0.25$$

Example 1 (b)

25 % of the families in a village have a subscription to a local newspaper
Define Z: $\sharp$ families we have to ask such that the last family we ask is the first one that has a subscription

$$P(Z = 1) = 0.25$$
$$P(Z = 2) = 0.75 \cdot 0.25$$
$$P(Z = 3) =$$

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25 % of the families in a village have a subscription to a local newspaper
Define Z: $\sharp$ families we have to ask such that the last family we ask is the first one that has a subscription

$$P(Z=1)=0.25$$
$$P(Z=2)=0.75\cdot 0.25$$
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$$P(Z=k) =$$

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Define Z: $\sharp$ families we have to ask such that the last family we ask is the first one that has a subscription

$$P(Z=1)=0.25$$
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$$P(Z=3)=0.75\cdot 0.75\cdot 0.25$$
$$P(Z=k)=0.75^{k-1}\cdot 0.25$$

Example 1 (b)

25 % of the families in a village have a subscription to a local newspaper
Define Z: ♯ families we have to ask such that the last family we ask is the first one that has a subscription

$$
\begin{aligned}
P(Z=1) &= 0.25 \\
P(Z=2) &= 0.75 \cdot 0.25 \\
P(Z=3) &= 0.75 \cdot 0.75 \cdot 0.25 \\
P(Z=k) &= 0.75^{k-1} \cdot 0.25
\end{aligned}
$$

If we change 0.25 into a $p \in [0,1]$, then $P(Z=k) = (1-p)^{k-1} \cdot p$

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Define Z: $\sharp$ families we have to ask such that the last family we ask is the first one that has a subscription

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P(Z = 1) &= 0.25 \\
P(Z = 2) &= 0.75 \cdot 0.25 \\
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P(Z = k) &= 0.75^{k-1} \cdot 0.25
\end{aligned}
$$

If we change 0.25 into a $p \in [0, 1]$, then $P(Z = k) = (1 - p)^{k-1} \cdot p \Rightarrow$ Geometric distribution $G(p)$

Geometric Distribution $G(p)$

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If $X \sim G(p)$, then $P(X = k) = (1-p)^{k-1} \cdot p$

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Expectation of Geometric Distribution

Let $X \sim G(p)$. Derive $E(X)$:

Geometric Distribution $G(p)$

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Expectation of Geometric Distribution

Let $X \sim G(p)$. Derive $E(X)$:

$$E(X) = \sum_{k=1}^{\infty} k(1 - p)^{k-1} p$$

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Let $X \sim G(p)$. Derive $E(X)$:

$$E(X) = \sum_{k=1}^{\infty} k(1-p)^{k-1}p = p\sum_{k=1}^{\infty}(-\frac{\mathrm{d}}{\mathrm{d}p}(1-p)^k)$$

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Let $X \sim G(p)$. Derive $E(X)$:

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E(X) &= \sum_{k=1}^{\infty} k(1-p)^{k-1} p = p \sum_{k=1}^{\infty} \left(-\frac{\mathrm{d}}{\mathrm{d}p}(1-p)^k\right) \\
&= p\frac{\mathrm{d}}{\mathrm{d}p}\left(-\sum_{k=1}^{\infty}((1-p)^k)\right) = p\frac{\mathrm{d}}{\mathrm{d}p}\left(-\frac{1-p}{1-(1-p)}\right)
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&= p \frac{\mathrm{d}}{\mathrm{d}p}(\frac{p-1}{p})
\end{aligned}
$$

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&= p\frac{\mathrm{d}}{\mathrm{d}p}(\frac{p-1}{p}) = p\frac{1}{p^2}
\end{aligned}
$$

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$$
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&= p\frac{\mathrm{d}}{\mathrm{d}p}(\frac{p-1}{p}) = p\frac{1}{p^2} = \frac{1}{p}
\end{aligned}
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book: Theorem 5.4

Geometric distribution

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If we have a Bernoulli process (a number of independent trials, each with success probability p) and X is the RV that is the trial giving the first success, then $X \sim G(p)$ and

Geometric distribution

If we have a Bernoulli process (a number of independent trials, each with success probability p) and X is the RV that is the trial giving the first success, then $X \sim G(p)$ and

- $P(X = k) = (1 - p)^{k-1} \cdot p$
- $E(X) = \frac{1}{p}$
- $V(X) = \frac{1-p}{p^2}$