Lecture 11: Some Important Discrete Probability Distributions (Part 2)

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Statistics (MAT1003)

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Outline

book: Sections 5.4, 5.5

And now . . .

1. **Last time**
   - Binomial Distribution
   - Multinomial Distribution
   - Hypergeometric Distribution
   - Exercises 5.1, 5.3, 5.7, 5.9, 5.11. 5.13, 5.17, 5. 23 (pp. 150–152)
2. **Hypergeometric Distribution**
3. **Geometric Distribution**
4. **Negative Binomial Distribution**
5. **The Poisson Process & Distribution**
   - What is Poisson Process?
   - Poisson Distribution

Binomial Distribution

Binomial distribution

If we do n independent trials, where each time the success probability is $p \in [0,1]$, and we define X to be the number of successes, then $X \sim B(n,p)$, and hence

$$P(X=k) = \underbrace{\binom{n}{k}}_{\text{pick } k \text{ out of } n} \cdot \underbrace{p^k}_{k \text{ successes, each with probability } p} \cdot \underbrace{(1-p)^{n-k}}_{n-k \text{ failures}}, \; k = 0,1,2,\ldots,n$$

Moreover, $X \sim B(n,p) \Rightarrow E(X) = np$, $V(X) = np(1-p)$

Multinomial distribution

If we have more than 2 possible outcomes, we call the distribution multinomial.
Notation $X \sim M(n; p_1, p_2, \ldots p_m)$
Computation:

$$P(x_1 = k_1, x_2 = k_2, \ldots, x_m = k_m) = \frac{n!}{k_1! \cdot k_2! \ldots \cdot k_m!} \cdot p_1^{k_1} \cdot p_2^{k_2} \cdot \ldots \cdot p_m^{k_m}$$

with $\sum_{i=1}^{m} p_i = 1$, $\sum_{j=1}^{m} k_j = n$

Example 2(d)

A bag contains 2 red, 3 blue, and 5 black marbles. What if we take the marbles without replacement? Let's pick 4

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$$P(X + Y = 3) =$$

$$P(X = 2, Y = 1) =$$

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Corresponding distribution: Hypergeometric (HW: derive general formula + try odd exercises from Sections 5.3 & 5.4)

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$$P(X = k) = \frac{\begin{pmatrix} s \\ k \end{pmatrix} \cdot \begin{pmatrix} N - s \\ n - k \end{pmatrix}}{\begin{pmatrix} N \\ n \end{pmatrix}}$$

where $\max\{0, n - (N - s)\} \leq k \leq \min\{n, s\}$
Also, $E(X) = \mu_X = \frac{n\,k}{N}$, $V(X) = \frac{N-n}{N-1} \cdot n \cdot \frac{k}{N} \cdot (1 - \frac{k}{N})$

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$$P(Z = 1) = 0.25$$
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$$P(Z = 2) = 0.75 \cdot 0.25$$

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$$V(X) = E(X^2) - \mu_X^2$$

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&= p \frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{p-1}{p}\right) = p \frac{1}{p^2} = \frac{1}{p} \\
V(X) &= E(X^2) - \mu_X^2 = \frac{1-p}{p^2}
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book: Theorem 5.3

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Exercise 5.55 (pp. 165)

And now . . .

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- Therefore

$$P(Y = y) = \binom{y-1}{k-1} p^{k-1} (1-p)^{y-k} \cdot p,$$

$y = k, k + 1, \ldots,$

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- $V(Y) = k \cdot \frac{1-p}{p^2}$ (trials are independent)

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If we have throw a die repeatedly, how many times does it take on average until 6 shows up for the 15^{th} time?

Y : $\sharp$ trials required, $Y \sim NB(15, \frac{1}{6})$

$E(Y) = \frac{k}{p} = \frac{15}{\frac{1}{6}} = 90 = 15 \cdot E(X)$

And now . . .

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Book: Section 5.5

Properties of Poisson Process

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The number X of outcomes occurring during a Poisson experiment is called a Poisson random variable, and its probability distribution is called the Poisson distribution.

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(Exercise from Calculus: Use ∞-degree of Taylor polynomial to show that $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$)

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Now: Odd exercises, Section 5.4