Lecture 13: Some Important Continuous Probability Distributions (Part 2)

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Statistics (MAT1003)

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Outline

book: Sections 6.2-6.4,6.6

And now . . .

Example 3: Estimation in a uniform distribution $U(0, L)$

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- We call $X_1, X_2, \ldots, X_{100}$ a random sample
- After 100 drawings we have 100 realizations, denoted by $x_1, x_2, \ldots, x_{100}$

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- As an estimate for L we now define the RV $B = \frac{101}{100} \cdot Z$. We have $E(B) = L$
- B is called an unbiased estimator for L

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Observation 1

- $E(\bar{X}) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \frac{1}{n}\sum_{i=1}^{n} \mu_X = \frac{1}{n}\cdot n\cdot \mu_X = \mu_X$

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Notice that $\underbrace{V(\bar{X}) = E\left\{(X-\mu)^2\right\}}_{\text{variance of sample mean}} \neq \underbrace{\bar{S}^2 \approx E\left\{(X_i - \bar{X})^2\right\}}_{\text{sample variance}}$

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Moreover, Observation 2 is independent of the actual distribution of the X_i

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    HW: What is the probability distribution of $\sum_{i=1}^n X_i$?

And now . . .

1. Flavor of estimation problems . . .
2. **Exponential Distribution**
   - Formulation
   - Expectation etc.
   - Application of the Exponential distribution
3. Normal Distribution
   - Basics
   - Examples
4. Exercises
5. Monday

Formulation

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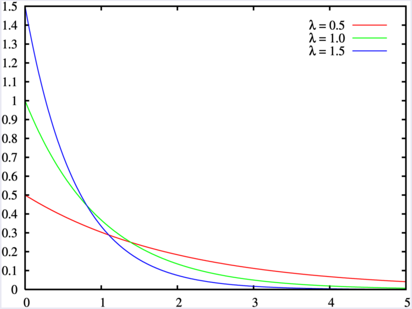

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$$\stackrel{bp}{=} [-x \cdot \exp(-\lambda\, x)]_{x=0}^{\infty} + \int_0^\infty \exp(-\lambda\, x) \mathrm{d}x$$

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&\overset{bp}{=} [-x \cdot \exp(-\lambda\, x)]_{x=0}^{\infty} + \int_0^\infty \exp(-\lambda\, x)\mathrm{d}x \\
&= -0 + 0 + \left[-\frac{1}{\lambda} \cdot \exp(-\lambda\, x)\right]_{x=0}^{\infty}
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F(x) &= 1 - \exp(-\lambda\, x) \quad \text{for } x > 0 \text{ and } 0 \text{ elsewhere} \\
E(X^2) &= \frac{2}{\lambda^2}
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V(X) &= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}
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Application of the Exponential distribution

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Suppose on average 6 people call some service number per minute. What is the probability that:

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(a) $X = \sharp$ calls in 3 minutes $\Rightarrow X \sim \mathcal{P}(18)$ minutes (time unit: 3 minutes here)

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(Similarly with $\lambda = 3 : P(Y \geq 1) = e^{-3}$)

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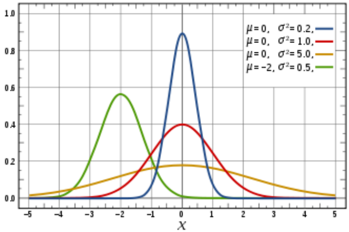

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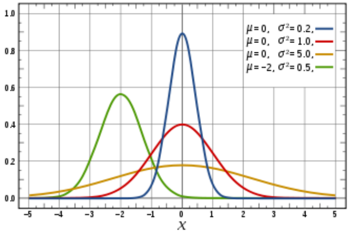

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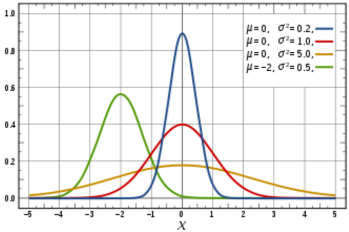

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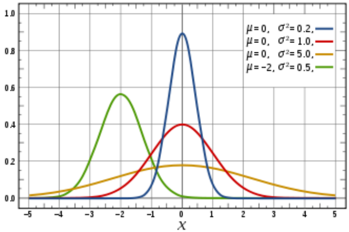

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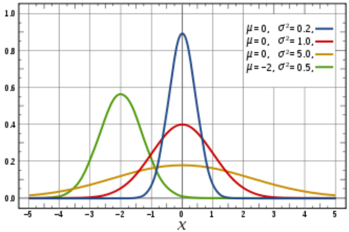

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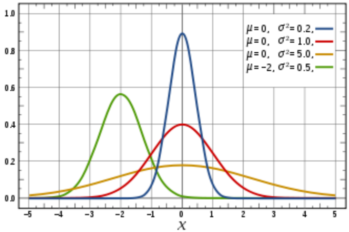

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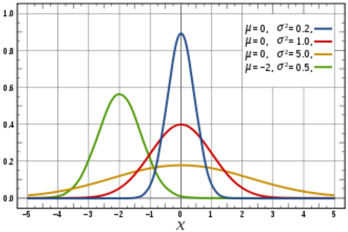

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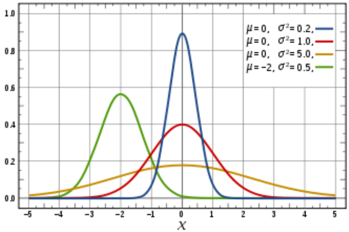

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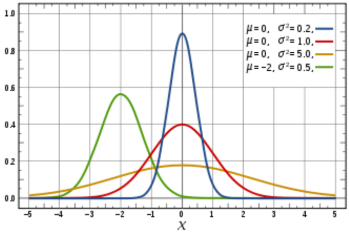

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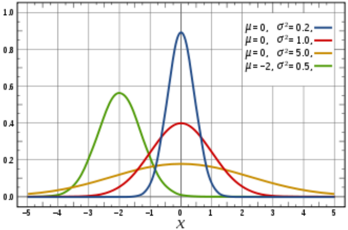

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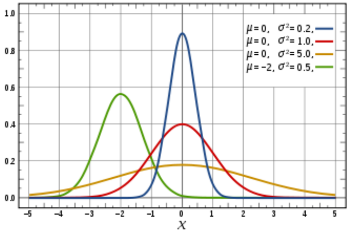

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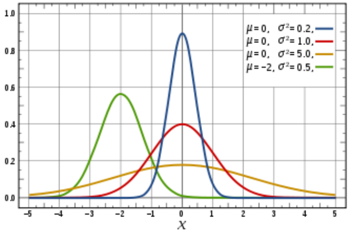

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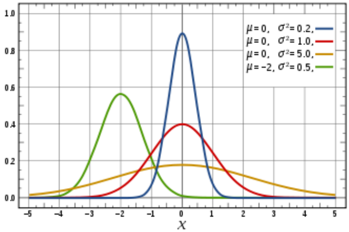

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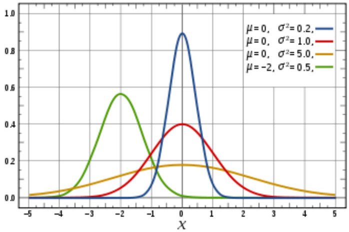

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- If $X_i \sim \mathcal{N}(\mu, \sigma)$, $i = 1, \ldots, n$, $X_1, \ldots, X_n$ IID. Then
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Example 6(a)

The net weight of a pack of coffee (500 grams) is a normally distributed RV with parameters $\mu = 505$ g and $\sigma = 5$ g. What is the probability that the net weight of a pack is at least 500 g?

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$$= 1 - P(Z \leq -1) \overset{TableA3}{=} 1 - 0.1587 = 0.8413$$

And now . . .

Computing together:

- book (pp. 164-165): 5.51, 5.59, 5.65
- book (pp. 186-187): 6.5, 6.7, 6.13

And now . . .

- Finishing up continuous PD's, introducing
  - Erlang distribution
  - Gamma-distribution
  - Chi-squared distribution
- Central limit theorem