Lecture 13: Some Important Continuous Probability Distributions (Part 2)

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Statistics (MAT1003)

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Outline

book: Sections 6.2-6.4,6.6

And now . . .

1. **Erlang distribution**
   - Formulation
   - Application of Erlang distribution
   - Gamma distribution
2. **Various Exercises**
3. **Chi-squared distribution**
   - Basics
   - Applications
   - Examples

Formulation

What is Erlang PDF?

Let $X_1, X_2, \ldots X_n$ be IID exponentially distributed RV's with parameter λ. Then the RV $X \stackrel{\text{def}}{=} X_1 + X_2 + \ldots + X_n$ has an Erlang distribution with parameters n and λ, $X \sim \mathrm{Erl}(n, \lambda)$,

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$$f(x) = \begin{cases} \frac{\lambda^n x^{n-1}}{(n-1)!} \exp(-\lambda x), & x > 0, \\ 0, & \text{elsewhere} \end{cases}$$

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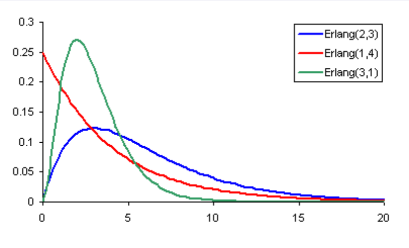

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- $Z = 3$ interarrival times
- $Z \sim \mathrm{Erl}(3, 18)$

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$$
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P(Z \geq \tfrac{1}{3}) &= \int_{\frac{1}{3}}^{\infty} \frac{\lambda^n x^{n-1}}{(n-1)!} \exp(-\lambda x)\, \mathrm{d}x = \int_{\frac{1}{3}}^{\infty} \frac{18^3 \cdot x^2}{2!} \exp(-18x)\, \mathrm{d}x \\
&= \left[\frac{18^3}{2} \cdot \left(-\frac{1}{18}\right) x^2 \exp(-18x) \right]_{x=\frac{1}{3}}^{\infty} \\
&+ \frac{18^3}{2} \int_{\frac{1}{3}}^{\infty} \frac{2}{18}\, x \exp(-18x)\, \mathrm{d}x = \left[-162\, x^2 \exp(-18x) \right]_{x=\frac{1}{3}}^{\infty}
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$$= [18 \exp(-6) + 6 \exp(-6) - \exp(-18\,x)]_{x=\frac{1}{3}}^{\infty}$$

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(Plug in $n = 1$ or $\alpha = 1$ to obtain the exponential distribution)

And now ...

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$$P(\frac{510 - 515}{\sqrt{26}} \leq Z \leq \frac{520 - 515}{\sqrt{26}})$$

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$$= 0.6730$$

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$$P(5120 \leq B \leq 5170) = P\left(\frac{5120 - 5150}{\sqrt{260}} \leq Z \leq \frac{5170 - 5150}{\sqrt{260}}\right)$$

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$$P(5120 \leq B \leq 5170) = P\left(\frac{5120 - 5150}{\sqrt{260}} \leq Z \leq \frac{5170 - 5150}{\sqrt{260}}\right)$$
$$= P\left(-1.86 \leq Z \leq 1.24\right)$$

Example 6 (from last Thursday)

The net weight of a pack of coffee (500 grams) is a normally distributed RV with parameters $\mu = 505$ grams and $\sigma = 5$ grams.

(c) What is the probability that the gross weight of 10 packs is between 5120 and 5170 grams? B : Gross weight of 10 packs,
$B = \sum_{i=1}^{10}(X_i + Y_i)$, with $X_i \sim \mathcal{N}(505, 5)$, $Y_i \sim \mathcal{N}(10, 1)$
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$$= 0.8925 - 0.0314 = 0.8611$$

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- $X \sim \mathcal{N}(\mu, \sigma)$ and $P(X \leq x) = p$, $P\left(Z \leq z = \frac{x-\mu}{\sigma}\right) = p$

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In general . . .

- $X \sim \mathcal{N}(\mu, \sigma)$ and $P(X \leq x) = p$, $P\left(Z \leq z = \frac{x-\mu}{\sigma}\right) = p$
- Look up z in table A3 and then calculate $x = \sigma\, x + \mu$

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Hence $X_1 - X_2 \sim \mathcal{N}(0, \sqrt{\sigma_{X_1}^2 + \sigma_{X_2}^2})$

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$P(Z \le 0.28) - P(Z \le -0.28) = 0.6103 - 0.3897 = 0.2206$

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  $= \frac{1}{2}x^{-\frac{1}{2}} \cdot \frac{1}{\sqrt{2\,\pi}} \exp(-\frac{x}{2}) + \frac{1}{2}x^{-\frac{1}{2}} \cdot \frac{1}{\sqrt{2\,\pi}} \exp(-\frac{x}{2}) = \frac{x^{-\frac{1}{2}} \exp(-\frac{x}{2})}{4\sqrt{\pi}}$
- Apparently: $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and $X = Z^2$ has a Gamma-distribution with parameters $\lambda = \frac{1}{2}$ ($\beta = 2$) and $\alpha = \frac{1}{2}$

And now ...

Formulation

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If $X_1, X_2, \ldots, X_n$ are IID, $X_i \sim \mathcal{N}(0, 1)$, then $\underline{\chi}^2 \stackrel{\text{def}}{=} X_1^2 + X_2^2 + \ldots + X_n^2$

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Book: Table A5: X s.t. $P(\underline{\chi}^2 \geq x) = \alpha$ for several values of v ($\sharp$ degrees of freedom) and α

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$\bar{X} = \frac{1}{5}(3.5 + 5.7 + 1.2 + 6.8 + 7.1) = 4.86$

$(n-1)s^2 = \sum_{i=1}^{n}(x_i - \bar{x})^2 = 24.732$

$\Rightarrow \underline{\chi}^2 = \frac{24.732}{\sigma^2}$

- Construct a 95% confidence interval for σ :
- Table A5: $P(\underline{\chi}^2 \geq 11.143) = 0.025$ and $P(\underline{\chi}^2 \geq 0.484) = 0.975$
  $\Rightarrow P(0.484 \leq 24.732/\sigma^2 \leq 11.143) = 0.95$
  $\Rightarrow P(24.732/11.143 \leq \sigma^2 \leq 24.732/0.484) = 0.95$

95% CI for σ^2: $[24.732/11.143, 24.732/0.484] = [0.220, 51.0999]$

Example 7 (b)

Let $X_1, X_2, \ldots, X_n \sim \mathcal{N}(\mu, \sigma)$ IDD with $\mu = 5$ and unknown σ. The realizations for $X_1, \ldots, X_5$ are 3.5, 5.7, 1.2, 6.8, and 7.1. What values of σ are reasonable

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95% CI for σ: $[1.490, 7.148]$

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Important!: These calculations use the fact that X_is are normally distributed. If it is not the case, we cannot use χ^2-distributions!