Lecture 15: Central Limit Theorem

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Statistics (MAT1003)

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Outline

1. **Chi-squared distribution (from last lecture)**
   - Basics
   - Applications
   - Examples
2. **Revision: Estimation Problems**
3. **Central Limit Theorem**
   - Confidence interval (CI)
   - Estimation of μ and σ from the same data
   - Theorem

book: Sections 6.2-6.4,6.6

And now . . .

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Basics

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book: Table A5 - X such that $P(\underline{\chi}^2 \geq x) = \alpha$ for several values of degree of freedom n (book: n denoted by v) and α

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$\Rightarrow Z_i^2 = \frac{(X_i-\mu)^2}{\sigma^2} \sim \chi_1^2$
$\Rightarrow \underline{\chi}^2 \stackrel{\text{def}}{=} \sum_{i=1}^n Z_i^2 = \frac{1}{\sigma^2} \sum_{i=1}^n (X_i - \mu)^2 \sim \chi_n^2$

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Let $X_1, X_2, \ldots, X_n \sim \mathcal{N}(\mu, \sigma)$ IDD with $\mu = 5$ and unknown σ. The realizations for $X_1, \ldots, X_5$ are 3.5, 5.7, 1.2, 6.8, and 7.1. What values of σ are reasonable?

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- So (fill in the realization for $\underline{\chi}^2$): $P(0.831 \leq \frac{24.83}{\sigma^2} \leq 12.832) = 0.975 - 0.025 = 0.95 \Leftrightarrow P(\sigma^2 \in [1.935, 29.88]) = 0.95$

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Similarly, $[1.935, 29.88]$ is a 95 % confidence interval for σ^2

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Let $\underline{\chi}^2 \stackrel{\text{def}}{=} \frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1} = \chi^2_4$

$\bar{x} = \frac{1}{5}(3.5 + 5.7 + 1.2 + 6.8 + 7.1) = 4.86$

$(n-1)s^2 = \sum_{i=1}^{n}(x_i - \bar{x})^2 = 24.732$

$\Rightarrow \underline{\chi}^2 = \frac{24.732}{\sigma^2}$

- Construct a 95% confidence interval for σ :
- Table A5: $P(\underline{\chi}^2 \geq 11.143) = 0.025$ and $P(\underline{\chi}^2 \geq 0.484) = 0.975$
  $\Rightarrow P(0.484 \leq 24.732/\sigma^2 \leq 11.143) = 0.95$
  $\Rightarrow P(24.732/11.143 \leq \sigma^2 \leq 24.732/0.484) = 0.95$

95% CI for σ^2: $[24.732/11.143, 24.732/0.484] = [0.220, 51.0999]$

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Important!: These calculations use the fact that X_i's are normally distributed. If it is not the case, we cannot use χ^2-distributions!

And now . . .

1. Chi-squared distribution (from last lecture)
   - Basics
   - Applications
   - Examples
2. **Revision: Estimation Problems**
3. Central Limit Theorem
   - Confidence interval (CI)
   - Estimation of μ and σ from the same data
   - Theorem

$X_1, X_2, \ldots, X_n$, IID $\mathcal{N}(\mu, \sigma)$

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- μ unknown: $\underline{\chi}^2 \stackrel{\text{def}}{=} \sum_{i=1}^{n} \frac{(X_i-\overline{X})}{\sigma^2} = \frac{(n-1)S^2}{\sigma^2} \sim \chi_{n-1}^2$ (chi-squared distribution with n degrees of freedom) $\Rightarrow$ calculate confidence interval (CI) for σ^2 and σ^2 (book: Section 9.12)

And now . . .

Confidence interval (CI)

Estimation of μ

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$X_1, X_2, \ldots, X_n \sim \mathcal{N}(\mu, \sigma)$ IDD, with σ known. Define $Z \stackrel{\text{def}}{=} \frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \sim \mathcal{N}(0,1)$

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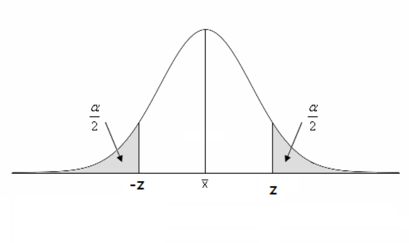

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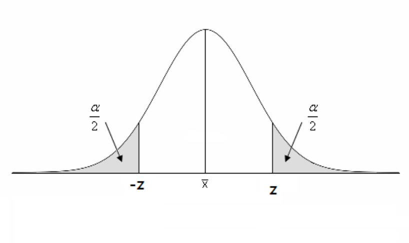

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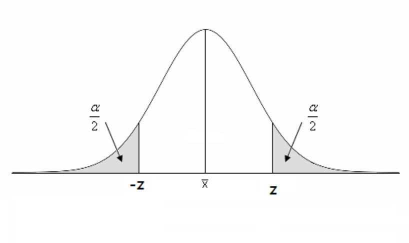

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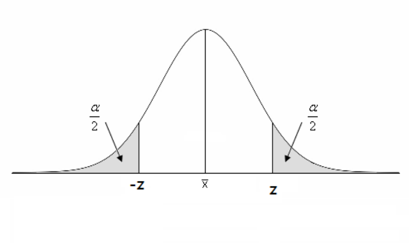

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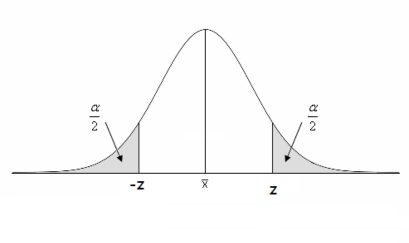

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- Hence

$$\left.\begin{array}{r}(1-\alpha)\mathrm{CI} \\ \text{or } (1-\alpha)\cdot 100\%\,\mathrm{CI}\end{array}\right\} \quad \text{for} \quad \mu \in \left[\bar{x} - \frac{z\cdot\sigma}{\sqrt{n}}, \bar{x} + \frac{z\cdot\sigma}{\sqrt{n}}\right]$$

Confidence interval (CI)

Example 1

$X_1, \dots, X_5 \sim \mathcal{N}(\mu, \sigma)$ IID
$\{x_1, \dots, x_5\} = \{3.5, 5.7, 1.2, 6.8, 7.1\}$; μ unknown, $\sigma = 2$

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$\Leftrightarrow$ 92 % CI for μ : $\left[4.86 - \frac{1.75 \cdot 2}{\sqrt{5}}, 4.86 + \frac{1.75 \cdot 2}{\sqrt{5}}\right] = [3.295, 6.425]$

Confidence interval (CI)

Example 2(from before)

$X_1, \ldots, X_{75} \sim \mathcal{N}(\mu, \sigma)$ IID, $\sigma = 0.0015$, $\bar{x} = 0.310$ (realization of $\overline{X}$)

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$\Rightarrow$ 95 % CI for μ : $[0.3097, 0.3103]$

Estimation of μ and σ from the same data

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- Let $Z \stackrel{\text{def}}{=} \frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \sim \mathcal{N}(0,1)$, $\underline{\chi}^2 \stackrel{\text{def}}{=} \frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$

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How to find $P(T \geq t)$?

- book – table A4: Values of t such that $P(T \geq t) = \alpha$ - for several $\alpha < 0.5$ and $v = \sharp$ degrees of freedom

Estimation of μ if σ is unknown

- Natural statistics for $\mu : X_1, \dots, X_n \sim \mathcal{N}(\mu, \sigma)$ IID
- $T = \frac{\overline{X} - \mu}{\frac{S}{\sqrt{n}}}$ (use sample standard deviation instead of σ)
- $T = \frac{\overline{X} - \mu}{S/\sqrt{n}} = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \cdot \frac{\sigma}{S}$
- Let $Z \stackrel{\text{def}}{=} \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \sim \mathcal{N}(0, 1)$, $\underline{\chi}^2 \stackrel{\text{def}}{=} \frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$
- Then: $T = \frac{Z}{\sqrt{\underline{\chi}^2/(n-1)}}$ and is distributed according to the *t*-distribution with $n-1$ degrees of freedom

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Due to symmetry around $t = 0$ we have $P(T \leq -t) = P(T \geq t) = \alpha$ or similarly: $P(T \leq -t) = 1 - \alpha$

Estimation of μ and σ from the same data

Example 1 (from before)

Let $\{x_1, \ldots, x_5\} = \{3.5, 5.7, 1.2, 6.8, 7.1\}$. Suppose that σ is unknown, $X_i \sim \mathcal{N}(\mu, \sigma)$. Calculate a 98.5% CI for μ.
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- Hence: $P(-4.088 \leq \frac{4.86-\mu}{2.487/\sqrt{5}} \leq 4.088) = 0.985$
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- Therefore: 98% CI for μ : $[0.313, 9.407]$

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General formula

If σ is unknown:

$$\left.\begin{array}{l} 1-\alpha \text{ CI for } \mu \\ (1-\alpha)\cdot 100\% \text{ CI for } \mu \end{array}\right\} : \left[\bar{x} - \frac{t \cdot s}{\sqrt{n}}, \bar{x} + \frac{t \cdot s}{\sqrt{n}}\right],$$

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- $T = \frac{\bar{x}-\mu}{S/\sqrt{n}} \sim t_{n-1} \Rightarrow \frac{0.310-\mu}{0.0028/\sqrt{95}} \sim t_{74}$ - not in the table, we approximate by t_{60} ("pretty close")

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Note (from before): $\underline{\chi}^2 \sim \chi^2_n$

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Note (from before): $\underline{\chi}^2 \sim \chi^2_n \Rightarrow E(\underline{\chi}^2) = n$, $V(\underline{\chi}^2) = 2\,n$

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- Let $X_1, \ldots, X_n$ be IID with unknown μ and σ.
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$$Z \stackrel{\text{def}}{=} \frac{\overline{X} - \mu}{\sigma / \sqrt{n}} \stackrel{\text{approx}}{\sim} \mathcal{N}(0, 1)$$

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Book: Sections 8.4 (σ known), 9.4 (σ unknown)

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Theorem

Central limit theorem (CLT)

- Let $X_1, \ldots, X_n$ be IID with unknown μ and σ.
- Then $E(\overline{X}) = \mu$, $V(\overline{X}) = \frac{\sigma^2}{n}$, and, if n is big enough,

$$Z \stackrel{\text{def}}{=} \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \approx \mathcal{N}(0, 1)$$

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$\Rightarrow$ $V(\frac{s^2}{\sigma^2}) = \frac{1}{(n-1)^2} \cdot V(\underline{\chi}^2) = \frac{2}{n-1} \stackrel{n\to\infty}{\to} 0$, whereas $E(\frac{s^2}{\sigma^2}) = \frac{n-1}{n-1} = 1 \Rightarrow$ That is why we may replace σ by S.